The No-Binding Regime of the Pauli-Fierz Model

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Abstract

The Pauli-Fierz model $H(\alpha)$ in nonrelativistic quantum electrodynamics is considered. The external potential V is sufficiently shallow and the dipole approximation is assumed. It is proven that there exist constants $0 < \alpha_- < \alpha_+$ such that $H(\alpha)$ has no ground state for $|\alpha| < \alpha_-$, which complements an earlier result stating that there is a ground state for $|\alpha| > \alpha_+$. We develop a suitable extension of the Birman-Schwinger argument. Moreover for any given $\delta > 0$ examples of potentials V are provided such that $\alpha_+ - \alpha_- < \delta$.

1 Introduction

Let us consider a quantum particle in an external potential described by the Schrödinger operator

$$H_{\mathrm{p}}(m) = -\frac{1}{2m}\Delta + V(x) \tag{1.1}$$

acting on $L^2(\mathbb{R}^d)$. If the potential V is short ranged and attractive and if the dimension $d \geq 3$, then there is a transition from unbinding to binding as the mass m is increased. More precisely, there is some critical mass, m_{c}, such that $H_{\mathrm{p}}(m)$ has no ground state for $0 < m < m_{\mathrm{c}}$ and a unique ground state for $m_{\mathrm{c}} < m$. In fact, the critical mass is given by

$$\frac{1}{2m_{\mathrm{c}}} = \left\| |V|^{1/2} (-\Delta)^{-1} |V|^{1/2} \right\| ,$$

see Lemma 3.3. We now couple $H_{\mathrm{p}}(m)$ to the quantized electromagnetic field with coupling strength $\alpha \geq 0$. The corresponding Hamiltonian is denoted by $H(\alpha)$. On a heuristic level, through the dressing by photons the particle becomes effectively more heavy, which means that the critical mass $c_0\alpha^2(\alpha)$ should be decreasing as a function of α with $m_{\mathrm{c}}(0) = m_{\mathrm{c}}$. In particular, if $m < m_{\mathrm{c}}$, then there should be an unbinding-binding transition as the coupling α is increased. This phenomenon has been baptized *enhanced binding* and has been studied for a variety of models by several authors [AK03, BV04, HVV03, HHS05, HS01, HS08]. In case $m > m_{\mathrm{c}}$ more general techniques are available and the existence of a unique ground state for the full Hamiltonian is proven in [AH97, BFS99, GLL01, LL03, Ger00, Spo98].

The heuristic picture also asserts that the full hamiltonian has a regime of couplings with no ground state. This property is more difficult to establish and the only result we are aware of is proved by Benguria and Vougalter [BV04]. In essence they establish that the line $m_{\mathrm{c}}(\alpha)$ is continuous as $\alpha \to 0$. (In fact, they use the strength of the potential as parameter). From this it follows that the no binding regime cannot be empty. In our paper, as in [HS01], we will use the dipole approximation for simplicity, but provide a fairly explicit bound on the critical mass. In the dipole approximation the effective mass $m_{\mathrm{eff}}(\alpha) = m + c_0\alpha^2$ with some explicitly computable coefficient c_0, see Eq. (2.10) below. Thus the most basic guess for $m_{\mathrm{c}}(\alpha)$ would be $m_{\mathrm{c}}(\alpha) + c_0\alpha^2 = m_{\mathrm{c}}$. The corresponding curve is displayed in Fig. 1. In fact the guess turns out to be a lower bound on the true $m_{\mathrm{c}}(\alpha)$. We will

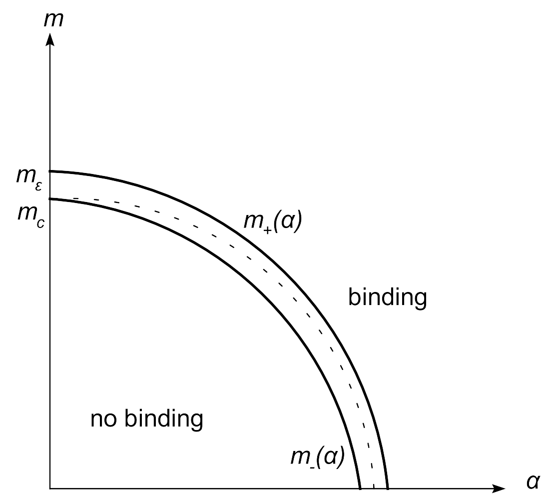

Figure 1: Upper and lower bounds on the critical mass $m_{\mathrm{c}}(\alpha)$. The dashed line indicates $m_{\mathrm{c}}(\alpha)$

complement our lower bound with an upper bound of the same qualitative form.

The unbinding for the Schrödinger operator $H_{\mathrm{p}}(m)$ is proven by the Birman-Schwinger principle. Formally one has

$$H_{\mathrm{p}}(m) = \frac{1}{2m}(-\Delta)^{1/2}\big(\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}\big)(-\Delta)^{1/2}.$$

If m is sufficiently small, then $2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ is a strict contraction. Hence the operator $\mathbb{1} + 2m(-\Delta)^{-1/2}V(-\Delta)^{-1/2}$ has a bounded inverse and $H_{\mathrm{p}}(m)$ has no eigenvalue in $(-\infty, 0]$. More precisely the Birman-Schwinger principle states that

$$\dim \mathbb{1}_{[\frac{1}{2m},\infty)}(V^{1/2}(-\Delta)^{-1}V^{1/2}) \geq \dim \mathbb{1}_{(-\infty,0]}(H_{\mathrm{p}}(m)). \tag{1.2}$$

For small m the left hand side equals 0 and thus $H_{\mathrm{p}}(m)$ has no eigenvalues in $(-\infty, 0]$.

Our approach will be to generalize (1.2) to the Pauli-Fierz model of non-relativistic quantum electrodynamics. The Pauli-Fierz Hamiltonian $H(\alpha)$ is defined on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F}$, where $\mathscr{F}$ denotes the boson Fock space. Transforming $H(\alpha)$ unitarily by U one arrives at

$$U^{-1}H(\alpha)U = H_0(\alpha) + W + g \tag{1.3}$$

as the sum of the free Hamiltonian

$$H_0(\alpha) = -\frac{1}{2m_{\text{eff}}(\alpha)}\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{f}}, \tag{1.4}$$

involving the effective mass of the dressed particle and the Hamiltonian H_{f} of the free boson field, the transformed interaction

$$W = T^{-1}(V \otimes \mathbb{1})T, \tag{1.5}$$

and the global energy shift g. $m_{\text{eff}}(\alpha)$ is an increasing function of α. We will show that (1.3) has no ground state for sufficiently small $|\alpha|$ by means of a Birman-Schwinger type argument such as (1.2). In combination with the results obtained in [HS01] we provide examples of external potentials V such that for some given $\delta > 0$ there exist two constants $0 < \alpha_- < \alpha_+$ satisfying

$$\delta > \alpha_+ - \alpha_- > 0 \tag{1.6}$$

and $H(\alpha)$ has no ground state for $|\alpha| < \alpha_-$ but has a ground state for $|\alpha| > \alpha_+$.

Our paper is organized as follows. In Section 2 we define the Pauli-Fierz model and in Section 3 we prove the absence of ground states. Section 4 lists examples of external potentials exhibiting the unbinding-binding transition.

2 The Pauli-Fierz Hamiltonian

We assume a space dimension $d \geq 3$ throughout, and take the natural unit: the velocity of light $c = 1$ and the Planck constant divided 2π, $\hbar = 1$. The Hilbert space $\mathscr{H}$ for the Pauli-Fierz Hamiltonian is given by

$$\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathscr{F},$$

where

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n (\oplus^{d-1} L^2(\mathbb{R}^d))\right]$$

denotes the boson Fock space over the $(d-1)$-fold direct sum $\oplus^{d-1} L^2(\mathbb{R}^d)$. Let $\Omega = \{1, 0, 0, ...\} \in \mathscr{F}$ denote the Fock vacuum. The creation operator and the annihilation operator are denoted by $a^*(f, j)$ and $a(f, j)$, $j = 1, \ldots, d-1$, $f \in L^2(\mathbb{R}^d)$, respectively, and they satisfy the canonical commutation relations

$$[a(f, j), a^*(g, j')] = \delta_{jj'}(f, g)\mathbb{1}, \quad [a(f, j), a(g, j')] = 0 = [a^*(f, j), a^*(g, j')]$$

with (f,g) the scalar product on $L^2(\mathbb{R}^d)$. We write

$$a^\sharp(f,j) = \int a^\sharp(k,j) f(k) dk, \quad a^\sharp = a, a^*, \tag{2.1}$$

The energy of a single photon with momentum $k \in \mathbb{R}^d$ is

$$\omega(k) = |k|. \tag{2.2}$$

The free Hamiltonian on $\mathscr{F}$ is then given by

$$H_{\mathrm{f}} = \sum_{j=1}^{d-1} \int \omega(k) a^*(k,j) a(k,j) dk. \tag{2.3}$$

Note that $\sigma(H_{\mathrm{f}}) = [0,\infty)$, and $\sigma_{\mathrm{p}}(H_{\mathrm{f}}) = \{0\}$. $\{0\}$ is a simple eigenvalue of H_{f} and $H_{\mathrm{f}}\Omega = 0$.

Next we introduce the quantized radiation field. The d-dimensional polarization vectors are denoted by $e_j(k) \in \mathbb{R}^d$, $j = 1, \ldots, d-1$, which satisfy $e_i(k) \cdot e_j(k) = \delta_{ij}$ and $e_j(k) \cdot k = 0$ almost everywhere on $\mathbb{R}^d$. The quantized vector potential then reads

$$A(x) = \sum_{j=1}^{d-1} \int \frac{1}{\sqrt{2\omega(k)}} e_j(k) \big(\hat{\varphi}(k) a^*(k,j) \mathrm{e}^{-ikx} + \hat{\varphi}(-k) a(k,j) \mathrm{e}^{ikx} \big) dk \tag{2.4}$$

for $x \in \mathbb{R}^d$ with ultraviolet cutoff $\hat{\varphi}$. Conditions imposed on $\hat{\varphi}$ will be supplied later. Assuming that V is centered, in the dipole approximation $A(x)$ is replaced by $A(0)$. We set $A = A(0)$. The Pauli-Fierz Hamiltonian $H(\alpha)$ in the dipole approximation is then given by

$$H(\alpha) = \frac{1}{2m} \left(p \otimes \mathbb{1} - \alpha \mathbb{1} \otimes A \right)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}, \tag{2.5}$$

where $\alpha \in \mathbb{R}$ is the coupling constant, V the external potential, and $p = (-i\partial_1, \ldots, -i\partial_d)$ the momentum operator. For notational convenience we omit the tensor notation $\otimes$ in what follows.

Assumption 2.1 *Suppose that V is relatively bounded with respect to $-\dfrac{1}{2m}\Delta$ with a relative bound strictly smaller than one, and*

$$\hat{\varphi}/\omega \in L^2(\mathbb{R}^d), \quad \sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^d). \tag{2.6}$$

By this assumption $H(\alpha)$ is self-adjoint on $D(-\Delta) \cap D(H_{\rm f})$ and bounded below for arbitrary $\alpha \in \mathbb{R}$ [Ara81, Ara83]. We need in addition some technical assumptions on $\hat{\varphi}$ which are introduced in [HS01, Definition 2.2]. We list them as

Assumption 2.2 *The ultraviolet cutoff $\hat{\varphi}$ satisfies (1)-(4) below.*

(1) $\hat{\varphi}/\omega^{3/2} \in L^2(\mathbb{R}^d)$*;*

(2) $\hat{\varphi}$ *is rotation invariant, i.e.* $\hat{\varphi}(k) = \chi(|k|)$ *with some real-valued function* χ *on* $[0, \infty)$*; and* $\rho(s) = |\chi(\sqrt{s})|^2 s^{(d-2)/2} \in L^\epsilon([0,\infty), ds)$ *for some* $1 < \epsilon$*,and there exists* $0 < \beta < 1$ *such that* $|\rho(s+h) - \rho(s)| \leq K|h|^\beta$ *for all* s *and* $0 < h \leq 1$ *with some constant* K*;*

(3) $\|\hat{\varphi}\omega^{(d-1)/2}\|_\infty < \infty$*;*

(4) $\hat{\varphi}(k) \neq 0$ *for* $k \neq 0$*.*

The Hamiltonian $H(\alpha)$ with $V = 0$ is quadratic and can therefore be diagonalized explicitly, which is carried out in [Ara83, HS01]. Assumption 2.2 ensures the existence of a unitary operator diagonalizing $H(\alpha)$.

Let

$$D_+(s) = m - \alpha^2 \frac{d-1}{d} \int \frac{|\hat{\varphi}(k)|^2}{s - \omega(k)^2 + i0} dk, \quad s \geq 0.$$

We see that $D_+(0) = m + \alpha^2 \frac{d-1}{d} \|\hat{\varphi}/\omega\|^2 > 0$ and the imaginary part of $D_+(s)$ is $\alpha^2 \frac{d-1}{d} \pi S_{d-1} \rho(s) \neq 0$ for $s \neq 0$, where ρ is defined in (2) of Assumption 2.2 and S_{d-1} is the volume of the $d-1$ dimensional unit sphere, and the real part of $D_+(s)$ satisfies that $\lim_{s\to\infty} \Re D_+(s) = m > 0$. These properties follows from Assumption 2.2. In particular

$$\inf_{s \geq 0} |D_+(s)| > 0. \tag{2.7}$$

Define

$$\Lambda_j^\mu(k) = \frac{e_j^\mu(k)\hat{\varphi}(k)}{\omega^{3/2}(k) D_+(\omega^2(k))}. \tag{2.8}$$

Then $\|\Lambda_j^\mu\| \leq C \|\hat{\varphi}/\omega^{3/2}\|$ for some constant C.

Proposition 2.3 *Under the assumptions 2.1 and 2.2, for each $\alpha \in \mathbb{R}$, there exist unitary operators U and T on $\mathscr{H}$ such that both map $D(-\Delta) \cap D(H_{\mathrm{f}})$ onto itself and*

$$U^{-1} H(\alpha) U = -\frac{1}{2m_{\mathrm{eff}}(\alpha)}\Delta + H_{\mathrm{f}} + T^{-1}VT + g, \tag{2.9}$$

where $m_{\mathrm{eff}}(\alpha)$ and g are constants given by

$$m_{\mathrm{eff}}(\alpha) = m + \alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/\omega\|^2, \tag{2.10}$$

$$g = \frac{d}{2\pi} \int_{-\infty}^{\infty} \frac{t^2\alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/(t^2+\omega^2)\|^2}{m + \alpha^2 \left(\frac{d-1}{d}\right) \|\hat{\varphi}/\sqrt{t^2+\omega^2}\|^2} dt. \tag{2.11}$$

Here U is defined in (4.29) of [HS01] and T by

$$T = \exp\left(-i\frac{\alpha}{m_{\mathrm{eff}}(\alpha)} p \cdot \phi\right), \tag{2.12}$$

where $\phi = (\phi_1, ..., \phi_d)$ is the vector field

$$\phi_\mu = \frac{1}{\sqrt{2}} \sum_{j=1}^{d-1} \int \left(\overline{\Lambda_j^\mu(k)} a^*(k,j) + \Lambda_j^\mu(k) a(k,j)\right) dk.$$

Proof: See [HS01, Appendix]. □

3 The Birman-Schwinger principle

3.1 The case of Schrödinger operators

Let $h_0 = -\frac{1}{2}\Delta$. We assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V is relatively form-bounded with respect to h_0 with relative bound $a < 1$, i.e., $D(|V|^{1/2}) \supset D(h_0^{1/2})$ and

$$|| V|^{1/2}\varphi\|^2 \le a\|h_0^{1/2}\varphi\|^2 + b\|\varphi\|^2, \quad \varphi \in D(h_0^{1/2}), \tag{3.1}$$

with some $b > 0$. Then the operators

$$R_E = (h_0 - E)^{-1/2} |V|^{1/2}, \qquad E < 0, \tag{3.2}$$

are densely defined. From (3.1) it follows that $R_E^* = |V|^{1/2}(h_0 - E)^{-1/2}$ is bounded and thus R_E is closable. We denote its closure by the same symbol. Let

$$K_E = R_E^* R_E. \tag{3.3}$$

Then K_E $(E < 0)$ is a bounded, positive self-adjoint operator and it holds

$$K_E f = |V|^{1/2} \, (h_0 - E)^{-1} \, |V|^{1/2} f, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Now let us consider the case $E = 0$. Let

$$R_0 = h_0^{-1/2} |V|^{1/2}. \tag{3.4}$$

The self-adjoint operator $h_0^{-1/2}$ has the integral kernel

$$h_0^{-1/2}(x, y) = \frac{a_d}{|x - y|^{d-1}}, \quad d \geq 3,$$

where $a_d = \sqrt{2}\pi^{(d-1)/2}/\Gamma((d-1)/2)$ and $\Gamma(\cdot)$ the Gamma function. It holds that

$$\left| (h_0^{-1/2} g, |V|^{1/2} f) \right| \leq a_d \|g\|_2 \| |V|^{1/2} f\|_{2d/(d+2)}$$

for $f, g \in C_0^\infty(\mathbb{R}^3)$ by the Hardy-Littlewood-Sobolev inequality. Since $f \in C_0^\infty(\mathbb{R}^3)$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, one concludes $\| |V|^{1/2} f\|_{2d/(d+2)} < \infty$. Thus $|V|^{1/2} f \in D(h_0^{-1/2})$ and R_0 is densely defined. Since V is relatively form-bounded with respect to h_0, R_0^* is also densely defined, and R_0 is closable. We denote the closure by the same symbol. We define

$$K_0 = R_0^* R_0. \tag{3.5}$$

Next let us introduce assumptions on the external potential V.

Assumption 3.1 *V satisfies that (1) $V \leq 0$ and (2) R_0 is compact.*

Lemma 3.2 *Suppose Assumption 3.1. Then*

(i) R_E, R_E^* and K_E $(E \leq 0)$ are compact.

(ii) $\|K_E\|$ is continuous and monotonously increasing in $E \leq 0$ and it holds that

$$\lim_{E \to -\infty} \|K_E\| = 0, \quad \lim_{E \uparrow 0} \|K_E\| = \|K_0\|. \tag{3.6}$$

Proof: Under (2) of Assumption 3.1, R_0^* and K_0 are compact. Since

$$(f, K_E f) \leq (f, K_0 f), \quad f \in C_0^\infty(\mathbb{R}^d), \tag{3.7}$$

extends to $f \in L^2(\mathbb{R}^3)$, K_E, R_E and R_E^* are also compact. Thus (i) is proven.

We will prove (ii). It is clear from (3.7) that K_E is monotonously increasing in E. Since R_0 is bounded, (3.7) holds on $L^2(\mathbb{R}^d)$ and

$$K_E = R_0^* \left((h_0 - E)^{-1} h_0\right) R_0, \quad E \leq 0. \tag{3.8}$$

From (3.8) one concludes that

$$\|K_E - K_{E'}\| \leq \|K_0\| \frac{|E - E'|}{|E'|}$$

for $E, E' < 0$. Hence $\|K_E\|$ is continuous in $E < 0$. We have to prove the left continuity at $E = 0$. Since $\|K_E\| \leq \|K_0\|$ $(E < 0)$, one has $\limsup_{E \uparrow 0} \|K_E\| \leq \|K_0\|$. By (3.8) we see that $K_0 = \text{s-}\lim_{E\uparrow 0} K_E$ and

$$\|K_0 f\| = \lim_{E\uparrow 0} \|K_E f\| \leq \left(\liminf_{E\uparrow 0} \|K_E\|\right) \|f\|, \quad f \in L^2(\mathbb{R}^d).$$

Hence we have $\|K_0\| \leq \liminf_{E\uparrow 0} \|K_E\|$ and $\lim_{E\uparrow 0} \|K_E\| = \|K_0\|$. It remains to prove that $\lim_{E\to-\infty} \|K_E\| = 0$. Since R_0^* is compact, for any $\epsilon > 0$, there exists a finite rank operator $T_\epsilon = \sum_{k=1}^n (\varphi_k, \cdot)\psi_k$ such that $n = n(\epsilon) < \infty$, $\varphi_k, \psi_k \in L^2(\mathbb{R}^d)$ and $\|R_0^* - T_\epsilon\| < \epsilon$. Then it holds that $\|K_E\| \leq (\epsilon + \|T_\epsilon h_0 (h_0 - E)^{-1}\|) \|R_0\|$. For any $f \in L^2(\mathbb{R}^d)$, we have

$$\|T_\epsilon h_0 (h_0 - E)^{-1} f\| \leq \left(\sum_{k=1}^n \|h_0 (h_0 - E)^{-1} \varphi_k\| \|\psi_k\|\right) \|f\|$$

and $\lim_{E\to-\infty} \|T_\epsilon h_0 (h_0 - E)^{-1}\| = 0$, which completes (ii). □

Let

$$H_{\mathrm{p}}(m) = -\frac{1}{2m}\Delta + V. \tag{3.9}$$

By (ii) of Lemma 3.2, we have $\lim_{E\to-\infty} \||V|^{1/2}(h_0 - E)^{-1/2}\| = 0$. Therefore V is infinitesimally form bounded with respect to h_0 and $H_{\mathrm{p}}(m)$ is the self-adjoint operator associated with the quadratic form

$$f, g \mapsto \frac{1}{m}(h_0^{1/2} f, h_0^{1/2} g) + (|V|^{1/2} f, |V|^{1/2} g)$$

for $f, g \in D(h_0^{1/2})$. Note that the domain $D(H_{\rm p}(m))$ is independent of m.

Under (2) of Assumption 3.1, the essential spectrum of $H_{\rm p}(m)$ coincides with that of $-\frac{1}{2m}\Delta$, hence $\sigma_{\rm ess}(H_{\rm p}(m)) = [0, \infty)$. Next we will estimate the spectrum of $H_{\rm p}(m)$ contained in $(-\infty, 0]$. Let $1\!\!1_{(\mathcal{O})}(T)$, $\mathcal{O} \subset \mathbb{R}$, be the spectral resolution of self-adjoint operator T and set

$$N_{\mathcal{O}}(T) = \dim \operatorname{Ran} 1\!\!1_{\mathcal{O}}(T). \tag{3.10}$$

The Birman-Schwinger principle [Sim05] states that

$$\begin{aligned} (E<0) \quad & N_{(-\infty, \frac{E}{m}]}\left(H_{\rm p}(m)\right) = N_{[\frac{1}{m},\infty)}(K_E), \\ (E=0) \quad & N_{(-\infty,0]}\left(H_{\rm p}(m)\right) \leq N_{[\frac{1}{m},\infty)}(K_0). \end{aligned} \tag{3.11}$$

Now let us define the constant $m_{\rm c}$ by the inverse of the operator norm of K_0,

$$m_{\rm c} = \|K_0\|^{-1}. \tag{3.12}$$

Lemma 3.3 *Suppose Assumption 3.1.*

(1) If $m < m_{\rm c}$, then $N_{(-\infty,0]}(H_{\rm p}(m)) = 0$.

(2) If $m > m_{\rm c}$, then $N_{(-\infty,0]}(H_{\rm p}(m)) \geq 1$.

Proof: It is immediate to see (1) by the Birman-Schwinger principle (3.11). Suppose $m > m_{\rm c}$. Then, using the continuity and monotonicity of $E \to \|K\|$, see Lemma 3.2, there exists $\epsilon > 0$ such that $m_{\rm c} < \|K_{-\epsilon}\|^{-1} \leq m$. Since $K_{-\epsilon}$ is positive and compact, $\|K_{-\epsilon}\| \in \sigma_{\rm p}(K_{-\epsilon})$ follows and hence $N_{[\frac{1}{m},\infty)}(K_{-\epsilon}) \geq 1$. Therefore (2) follows again from the Birman-Schwinger principle. □

Remark 3.4 *By Lemma 3.3, the critical mass at zero coupling $m_{\rm c}(0) = m_{\rm c}$.*

In the case $m > m_{\rm c}$, by the proof of Lemma 3.3 one concludes that the bottom of the spectrum of $H_{\rm p}(m)$ is strictly negative. For $\epsilon > 0$ we set

$$m_\epsilon = \|K_{-\epsilon}\|^{-1}. \tag{3.13}$$

Corollary 3.5 *Suppose Assumption 3.1 and $m > m_\epsilon$. Then*

$$\inf \sigma\left(H_{\rm p}(m)\right) \leq \frac{-\epsilon}{m}. \tag{3.14}$$

Proof: The Birman-Schwinger principle states that $1 \leq N_{(-\infty, -\frac{\epsilon}{m}]}\left(H_{\rm p}(m)\right)$, since $1/m < \|K_{-\epsilon}\|$, which implies the corollary. □

3.2 The case of the Pauli-Fierz model

In this subsection we extend the Birman-Schwinger type estimate to the Pauli-Fierz Hamiltonian.

Lemma 3.6 *Suppose Assumption 3.1. If $m < m_{\rm c}$, then the zero coupling Hamiltonian $H_{\rm p}(m) + H_{\rm f}$ has no ground state.*

Proof: Since the Fock vacuum Ω is the ground state of $H_{\rm f}$, $H_{\rm p}(m) + H_{\rm f}$ has a ground state if and only if $H_{\rm p}(m)$ has a ground state. But $H_{\rm p}(m)$ has no ground state by Lemma 3.3. Therefore $H_{\rm p}(m) + H_{\rm f}$ has no ground state. □

From now on we discuss $U^{-1}H(\alpha)U$ with $\alpha \neq 0$. We set

$$U^{-1}H(\alpha)U = H_0(\alpha) + W + g, \tag{3.15}$$

where

$$\begin{aligned} H_0(\alpha) &= -\frac{1}{2m_{\rm eff}(\alpha)}\Delta + H_{\rm f}, \\ W &= T^{-1}VT. \end{aligned} \tag{3.16}$$

Theorem 3.7 *Suppose Assumptions 2.1, 2.2 and 3.1. If $m_{\rm eff}(\alpha) < m_{\rm c}$, then $H_0(\alpha) + W + g$ has no ground state.*

Proof: Since g is a constant, we prove the absence of ground state of $H_0(\alpha) + W$. Since V is negative, so is W. Hence $\inf \sigma(H_0(\alpha)+W) \leq \inf \sigma(H_0(\alpha)) = 0$. Then it suffices to show that $H_0(\alpha) + W$ has no eigenvalues in $(-\infty, 0]$. Let $E \in (-\infty, 0]$ and set

$$\mathcal{K}_E = |W|^{1/2}(H_0(\alpha) - E)^{-1}|W|^{1/2}, \tag{3.17}$$

where $|W|^{1/2}$ is defined by the functional calculus. We shall prove now that if $H_0(\alpha) + W$ has eigenvalue $E \in (-\infty, 0]$, then $\mathcal{K}_E$ has eigenvalue 1. Suppose that $(H_0(\alpha) + W - E)\varphi = 0$ and $\varphi \neq 0$, then

$$\mathcal{K}_E|W|^{1/2}\varphi = |W|^{1/2}\varphi.$$

Moreover if $|W|^{1/2}\varphi = 0$, then $W\varphi = 0$ and hence $(H_0(\alpha) - E)\varphi = 0$, but $H_0(\alpha)$ has no eigenvalue by Lemma 3.6. Then $|W|^{1/2}\varphi \neq 0$ is concluded and $\mathcal{K}_E$ has eigenvalue 1. Then it is sufficient to see $\|\mathcal{K}_E\| < 1$ to show that

$H_0(\alpha) + W$ has no eigenvalues in $(-\infty, 0]$. Notice that $-\frac{1}{2m_{\text{eff}}(\alpha)}\Delta$ and T commute, and

$$\left\| (-\Delta)^{1/2} (H_0(\alpha) - E)^{-1} (-\Delta)^{1/2} \right\| \leq 2m_{\text{eff}}(\alpha).$$

Then we have

$$\|\mathcal{K}_E\| \leq \left\| |V|^{1/2} \left(-\frac{1}{2m_{\text{eff}}(\alpha)} \Delta \right)^{-1/2} \right\|^2 = m_{\text{eff}}(\alpha) \|K_0\| = \frac{m_{\text{eff}}(\alpha)}{m_{\text{c}}} < 1$$

and the proof is complete. $\square$

4 Absence and existence of a ground state

In this section we establish the absence, resp. existence, of a ground state of the Pauli-Fierz Hamiltonian $H_0(\alpha) + W$. Let $\kappa > 0$ be a parameter and let us define the Pauli-Fierz Hamiltonian with scaled external potential $V_\kappa(x) = V(x/\kappa)/\kappa^2$ by

$$H_\kappa = \frac{1}{2m}(p - \alpha A)^2 + V_\kappa + H_{\text{f}}. \tag{4.1}$$

We also define K_κ by $H(\alpha)$ with $a^\sharp$ replaced by $\kappa a^\sharp$. Then

$$K_\kappa = \frac{1}{2m}(p - \kappa\alpha A)^2 + V + \kappa^2 H_{\text{f}}. \tag{4.2}$$

H_κ and $\kappa^{-2} K_\kappa$ are unitarily equivalent,

$$H_\kappa \cong \kappa^{-2} K_\kappa. \tag{4.3}$$

Let $m < m_{\text{c}}$ and $\epsilon > 0$. We define the function

$$\alpha_\epsilon = \left(\frac{d-1}{d} \|\hat{\varphi}/\omega\|^2\right)^{-1/2} \sqrt{m_\epsilon - m}, \quad \epsilon > 0 \tag{4.4}$$

$$\alpha_0 = \left(\frac{d-1}{d} \|\hat{\varphi}/\omega\|^2\right)^{-1/2} \sqrt{m_{\text{c}} - m}, \tag{4.5}$$

where we recall that $m_\epsilon = \|K_{-\epsilon}\|^{-1}$ for $\epsilon \geq 0$. Note that

(1) $|\alpha| < \alpha_0$ if and only if $m_{\text{eff}}(\alpha) < m_{\text{c}}$;

(2) $|\alpha| > \alpha_\epsilon$ if and only if $m_{\rm eff}(\alpha) > m_\epsilon$.

Note that $\alpha_0 < \alpha_\epsilon$ because of $m_\epsilon > m_{\rm c}$. Since $\lim_{\epsilon\downarrow 0} m_\epsilon = m_{\rm c}$, it holds that $\lim_{\epsilon\downarrow 0}\alpha_\epsilon = \alpha_0$. We furthermore introduce assumptions on the external potential V and ultraviolet cutoff $\hat{\varphi}$.

Assumption 4.1 *The external potential V and the ultraviolet cutoff $\hat{\varphi}$ satisfies:*

(1) $V \in C^1(\mathbb{R}^d)$ and $\nabla V \in L^\infty(\mathbb{R}^d)$;

(2) $\hat{\varphi}/\omega^{5/2} \in L^2(\mathbb{R}^d)$.

We briefly comment on (1) of Assumption 4.1. We know that

$$H_0(\alpha) + W = -\frac{1}{2m_{\rm eff}(\alpha)}\Delta + V + H_{\rm f} + V(\cdot - \frac{\alpha}{m_{\rm eff}(\alpha)}\phi) - V.$$

The term on the right-hand side above, $H_{\rm int} = V(\cdot - \frac{\alpha}{m_{\rm eff}(\alpha)}\phi) - V$, is regarded as the interaction, and

$$H_{\rm int} \sim \frac{\alpha}{m_{\rm eff}(\alpha)}\nabla V(\cdot)\cdot\phi.$$

By (1) of Assumption 4.1, we have

$$\|H_{\rm int}\Phi\| \le C\|(H_{\rm f}+1)^{1/2}\Phi\|$$

with some constant C independent of α. This estimate follows from the fundamental inequality $\|a^\sharp(f)\Phi\| \le \|f/\sqrt{\omega}\|\|(H_{\rm f}+1)^{1/2}\Phi\|$. Then the interaction has a uniform bound with respect to the coupling constant α. Since the decoupled Hamiltonian $-\frac{1}{2m_{\rm eff}(\alpha)}\Delta + V + H_{\rm f}$ has a ground state for sufficiently large α, it is expected that $H_0(\alpha) + W$ also has a ground state for sufficiently large α. This is rigorously proven in (1) of Theorem 4.2 below. Now we are in the position to state the main theorem.

Theorem 4.2 *Suppose Assumptions 2.1, 2.2, 3.1 and 4.1. Then (1) and (2) below hold.*

(1) For any $\epsilon > 0$, there exists κ_ϵ such that for all $\kappa > \kappa_\epsilon$, H_κ has a unique ground state for all α such that $|\alpha| > \alpha_\epsilon$,

(2) H_κ has no ground state for all $\kappa > 0$ and all α such that $|\alpha| < \alpha_0$.

Proof: Let U_κ (resp. T_κ) be defined by U (resp. T) with ω and $\hat{\varphi}$ replaced by $\kappa^2\omega$ and $\kappa\hat{\varphi}$. Then

$$U_\kappa^{-1}K_\kappa U_\kappa = H_{\rm p}(m_{\rm eff}(\alpha)) + \kappa^2 H_{\rm f} + \delta V_\kappa + g, \tag{4.6}$$

where $\delta V_\kappa = T_\kappa^{-1} V T_\kappa - V$. Note that g is independent of κ. Since $U_\kappa^{-1}K_\kappa U_\kappa$ is unitary equivalent to $\kappa^2 H_\kappa$, we prove the existence of a ground state for $U_\kappa^{-1}K_\kappa U_\kappa$. Let $N = \sum_{j=1}^{d-1}\int a^*(k,j)a(k,j)dk$ be the number operator. Since $H_{\rm p}(m_{\rm eff}(\alpha))$ has a ground state by the assumption $|\alpha| > \alpha_\epsilon$, i.e., $m_{\rm eff}(\alpha) > m_{\rm c}$, it can be shown that $U_\kappa^{-1}K_\kappa U_\kappa + \nu N$ with $\nu > 0$ also has a ground state, see [HS01, p.1168] for details. We denote the normalized ground state of $U_\kappa^{-1}K_\kappa U_\kappa + \nu N$ by $\Psi_\nu = \Psi_\nu(\kappa)$. Since the unit ball in a Hilbert space is weakly compact, there exists a subsequence of $\Psi_{\nu'}$ such that the weak limit $\Psi = \lim_{\nu'\to 0}\Psi_{\nu'}$ exists. If $\Psi \neq 0$, then Ψ is a ground state [AH97]. Let $P = 1\!\!1_{[\Sigma,0)}(-\frac{1}{2m_{\rm eff}(\alpha)}\Delta + V)\otimes 1\!\!1_{\{0\}}(H_{\rm f})$ and $\Sigma = \inf\sigma(H_{\rm p}(m_{\rm eff}(\alpha)))$. Adopting the arguments in the proof of [HS01, Lemma 3.3], we conclude

$$(\Psi, P\Psi) \geq 1 - \frac{|\alpha|\varepsilon\|\hat{\varphi}/\omega^{5/2}\|^2}{\kappa^3 m_{\rm eff}(\alpha)} - \frac{\frac{3}{2}\frac{D}{\kappa}}{\kappa^2(|\Sigma| - \frac{3}{2}\frac{D}{\kappa})}, \tag{4.7}$$

where $\varepsilon > 0$ and D are constants independent of κ and α. Since $m_{\rm eff}(\alpha) > m_\epsilon > m_{\epsilon/2}$,

$$\Sigma \leq \inf\sigma(H_{\rm p}(m_\epsilon)) \leq -\frac{\epsilon}{2m_\epsilon} \tag{4.8}$$

by Corollary 3.5. By (4.8) and (4.7) we have

$$(\Psi, P\Psi) \geq \kappa^{-3}\left(\rho(\kappa) - \varepsilon\|\hat{\varphi}/\omega^{5/2}\|^2 \frac{|\alpha|}{m_{\rm eff}(\alpha)}\right), \tag{4.9}$$

where $\rho(\kappa) = \kappa^3 - \dfrac{\kappa}{\xi\kappa - 1}$ with $\xi = \frac{2\epsilon}{3m_\epsilon D}$. Then there exists $\kappa_\epsilon > 0$ such that the right-hand side of (4.9) is positive for all $\kappa > \kappa_\epsilon$ and all $\alpha \in \mathbb{R}$. Actually a sufficient condition for the positivity of the right-hand side of (4.9) is

$$\rho(\kappa) > \frac{\varepsilon\|\hat{\varphi}/\omega^{5/2}\|^2}{2\sqrt{m}\|\hat{\varphi}/\omega\|}, \tag{4.10}$$

since $\sup_\alpha \frac{|\alpha|}{m_{\rm eff}(\alpha)} = (2\sqrt{m}\|\hat{\varphi}/\omega\|)^{-1}$. Then $\Psi \neq 0$ for all $\kappa > \kappa_\epsilon$. Thus the ground state exists for all $|\alpha| > \alpha_\epsilon$ and all $\kappa > \kappa_\epsilon$ and (1) is complete.

We next show (2). Notice that

$$U_\kappa^{-1} H_\kappa U_\kappa = -\frac{1}{2m_{\rm eff}(\alpha)}\Delta + H_{\rm f} + T^{-1}V_\kappa T + g.$$

Define the unitary operator u_κ by $(u_\kappa f)(x) = k^{d/2} f(x/\kappa)$. Then we infer $V_\kappa = \kappa^{-2} u_\kappa V u_\kappa^{-1}$, $-\Delta = \kappa^{-2} u_\kappa(-\Delta) u_\kappa^{-1}$ and

$$\| |V_\kappa|^{1/2} (-\Delta)^{-1} |V_\kappa|^{1/2} \| = \kappa^{-2} \| u_\kappa |V|^{1/2} u_\kappa^{-1} (-\Delta)^{-1} u_\kappa |V|^{1/2} u_\kappa^{-1} \| = \|K_0\|.$$

(2) follows from Theorem 3.7. $\square$

Corollary 4.3 *Let arbitrary $\delta > 0$ be given. Then there exists an external potential $\tilde{V}$ and constants $\alpha_+ > \alpha_-$ such that*

(1) $0 < \alpha_+ - \alpha_- < \delta$*;*

(2) $H(\alpha)$ *has a ground state for* $|\alpha| > \alpha_+$ *but no ground state for* $|\alpha| < \alpha_-$*.*

Proof: Suppose that V satisfies Assumption 3.1. For $\delta > 0$ we take $\epsilon > 0$ such that $\alpha_\epsilon - \alpha_0 < \delta$. Take a sufficiently large κ such that (4.10) is fulfilled, and set $\tilde{V}(x) = V(x/\kappa)/\kappa^2$. Define $H(\alpha)$ by the Pauli-Fierz Hamiltonian with potential $\tilde{V}$. Then $H(\alpha)$ satisfies (1) and (2) with $\alpha_+ = \alpha_\epsilon$ and $\alpha_- = \alpha_0$. $\square$

Remark 4.4 (Upper and lower bound of $m_{\rm c}(\alpha)$) Corollary 4.3 implies the upper and lower bounds

$$\begin{array}{l} m_-(\alpha) \le m_{\rm c}(\alpha) \le m_+(\alpha), \\ m_{\rm c}(0) = m_{\rm c}, \end{array} \tag{4.11}$$

where

$$\begin{array}{l} m_-(\alpha) = m_0 - \alpha^2 \frac{d-1}{d} \|\hat{\varphi}/\omega\|^2, \\ m_+(\alpha) = m_\epsilon - \alpha^2 \frac{d-1}{d} \|\hat{\varphi}/\omega\|^2. \end{array}$$

Fix the coupling constant α. If $m < m_-(\alpha)$, then there is no ground state, and if $m > m_+(\alpha)$, then the ground state exists, compare with Fig. 1.

Remark 4.5 ($m_{\rm c}(\alpha)$ for sufficiently large α) Let $(\frac{d-1}{d}\|\hat{\varphi}/\omega\|^2)^{-1} m_\epsilon < \alpha^2$. Then by Remark 4.4, $H(\alpha)$ has a ground state for arbitrary $m > 0$. It is an open problem to establish whether this is an artifact of the dipole approximation or in fact holds also for the Pauli-Fierz operator.

5 Examples of external potentials

In this section we give examples of potentials V satisfying Assumption 3.1. The self-adjoint operator h_0^{-1} has the integral kernel

$$h_0^{-1}(x,y) = \frac{b_d}{|x-y|^{d-2}}, \quad d \geq 3,$$

with $b_d = 2\Gamma((d/2)-1)/\pi^{(d/2)-2}$. It holds that

$$(f, K_0 f) = \int dx \int dy \overline{f(x)} K_0(x,y) f(y), \tag{5.1}$$

where

$$K_0(x,y) = b_d \frac{|V(x)|^{1/2}|V(y)|^{1/2}}{|x-y|^{d-2}}, \quad d \geq 3, \tag{5.2}$$

is the integral kernel of operator K_0. We recall the Rollnik class $\mathscr{R}$ of potentials is defined by

$$\mathscr{R} = \left\{ V \Big| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \frac{|V(x)V(y)|}{|x-y|^2} < \infty \right\}.$$

By the Hardy-Littlewood-Sobolev inequality, $\mathscr{R} \supset L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ with $1/p + 1/r = 4/3$. In particular, $L^{3/2}(\mathbb{R}^3) \subset \mathscr{R}$.

Example 5.1 ($d = 3$ and Rollnik class) Let $d = 3$. Suppose that V is negative and $V \in \mathscr{R}$. Then $K_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Hence K_0 is Hilbert-Schmidt and Assumption 3.1 is satisfied.

The example can be extended to dimensions $d \geq 3$.

Example 5.2 ($d \geq 3$ and $V \in L^{d/2}(\mathbb{R}^d)$) Let $L^p_w(\mathbb{R}^d)$ be the set of Lebesgue measurable function u such that $\sup_{\beta>0} \beta \left|\{x \in \mathbb{R}^d | |u(x) > \beta\}\right|_L^{1/p} < \infty$, where $|E|_L$ denotes the Lebesgue measure of $E \subset \mathbb{R}^d$. Let $g \in L^p(\mathbb{R}^d)$ and $u \in L^p_w(\mathbb{R}^d)$ for $2 < p < \infty$. Define the operator $B_{u,g}$ by

$$B_{u,g}h = (2\pi)^{-d/2} \int e^{ikx} u(k) g(x) h(x) dx.$$

It is shown in [Cwi77, Theorem, p.97] that $B_{u,g}$ is a compact operator on $L^2(\mathbb{R}^d)$. It is known that $u(k) = 2|k|^{-1} \in L^d_w(\mathbb{R}^d)$ for $d \geq 3$. Let F denote Fourier transform on $L^2(\mathbb{R}^d)$, and suppose that $V \in L^{d/2}(\mathbb{R}^d)$. Then $B_{u,|V|^{1/2}}$ is compact on $L^2(\mathbb{R}^d)$ and then $R_0^* = F B_{u,V^{1/2}} F^{-1}$ is compact. Thus R_0 is also compact.

Assume that $V \in L^{d/2}(\mathbb{R}^d)$. Let us now see the critical mass of zero coupling $m_{\rm c} = m_0$. By the Hardy-Littlewood-Sobolev inequality, we have

$$|(f, K_0 f)| \le D_V \|f\|_2^2, \tag{5.3}$$

where

$$D_V = \sqrt{2}\pi \frac{\Gamma((d/2)-1)}{\Gamma((d/2)+1)} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{2/d} \|V\|_{d/2}^2, \tag{5.4}$$

a constant in (5.4) is proved by Lieb [Lie83]. Then

$$\|K_0\| \le D_V. \tag{5.5}$$

By (5.5) we have $m_{\rm c} \ge D_V^{-1}$. In particular in the case of $d = 3$,

$$m_{\rm c} \ge \frac{3}{\sqrt{2}\pi^{2/3} 4^{5/3}} \|V\|_{3/2}^{-2}. \tag{5.6}$$

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